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Conformal Field Theory Description of Highly Correlated States in Rapidly Rotating Bose Gases

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Takk

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Contents

Conventions and notations

- Quasiparticles

  The term "quasiparticle" is in this thesis taken to mean the opposite of a quasihole. This differs from much of the litterature, where "quasiparticle" is taken as a common term for quasielectrons and quasiholes.

- Vertex operators

  All vertex operators V, H and P in this thesis are implicitly normal ordered.

- Summation

  When greek indices are used, Einstein's summation convention is implied:

$$a_\mu b^\mu \equiv \sum_\mu a_\mu b_\mu$$

Chapter 1

Introduction

Interest in gaseous Bose-Einstein Condensates (BEC) has increased both theoretically and experimentally over the recent years. This phenomenon was first realized experimentally in 1995 by a group led by E. Cornell and C. Wiemann [1], and soon after by a group led by W. Ketterle [2]. For this discovery, the three were awarded the 2001 Nobel Prize in Physics.

When a quantum fluid such as a BEC is rotated, its behaviour is radically different from classical fluids. A classical fluid can sustain rigid body rotation, but rotating a BEC will instead generate quantized vortices. As the angular momentum is increased, more vortices will appear, arranged in a triangular lattice as shown in figure 1.1. Experiments with rotating Bose gases are reaching ever higher angular momenta [3–6], and are approaching the point where the vortex lattice is expected to melt [7]. The system is then expected to enter the so-called quantum Hall regime. This means that the system is mathematically equivalent to the quantum Hall effect (QHE), which makes it possible to utilize known techniques from this field on the rotating Bose gas [8]. One method that has proven fruitful in the QHE is to express trial wave functions as conformal field theory (CFT) correlators.

In this thesis we will utilize CFT correlators to propose trial wave functions

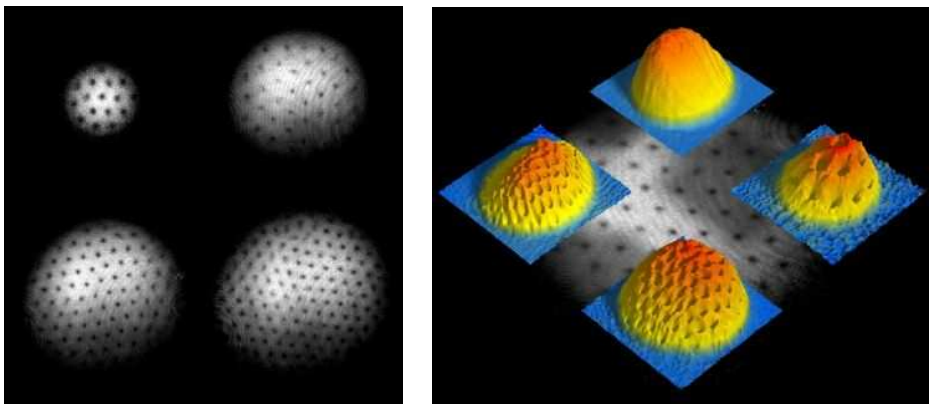

Figure 1.1: Triangular vortex lattice at different angular momenta. Images from Ketterle's group at MIT.

for higly correlated state in the rapidly rotating Bose gas. In particular we will reproduce the wave functions proposed by the composite fermion (CF) model. This model was originally constructed for the quantum Hall system [9], but it has also been used in context of the rotating Bose gas [10, 11].

In chapter 2 we will study the quantum Hall effect and the composite fermion model. We will see in chapter 3 how the problem of rotating Bose gases can be solved as a lowest Landau level problem, and how composite fermions may be formed in this system. Chapter 4 will describe the concept of CFT correlators and how they have been used to construct wave functions for the QHE. Lastly, in chapter 5 we will propose wavefunctions for several states predicted to emerge as the vortex lattice melts, and we will see how they compare to wavefunctions obtained by other methods.

Chapter 2

The Quantum Hall Effect

2.1 The Hall Effect

The Hall effect is a well-known phenomenon in classical electrodynamics. It is a consequence of the radial force exerted on moving charge carriers by a perpendicular magnetic field.

Consider a flat slab of a conducting material in the xy-plane, through which we send a current I in the x-direction. If we now impose a magnetic field B along the z-direction, this will push the charge carriers towards one side of the sample due to the Lorentz force $F_m = ev_xB$, where e is the elementary charge and v_x is the drift velocity of the free charge carriers. See figure 2.1. We can write the current density as $j_x = nev_x$, where n is the density of charge carriers. Combining these two gives

$$F_m = \frac{j_x B}{n}.$$

This force generates a charge gradient perpendicular to the current, and at equilibrium we can measure a voltage V_H along the y-direction, set up by the

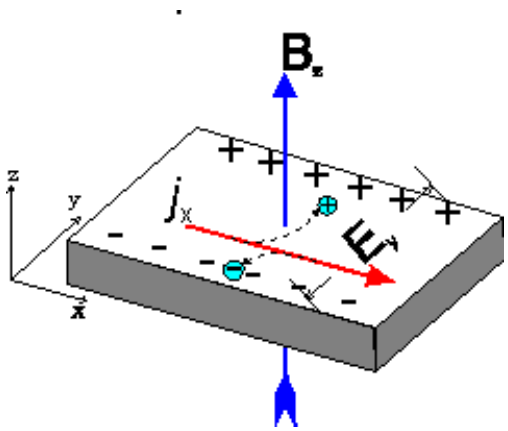

Figure 2.1: The classical Hall effect. The charge carriers are pushed to the side by the Lorentz force, setting up a field E_y across the conductor. [12]

electric field E_y across the conductor. This effect is called the Hall effect after its discoverer E. H. Hall, and the voltage V_H is called the Hall voltage. The electric force generated by V_H is

$$F_e = \frac{V_H e}{L}$$

where L is the width of the slab. Requiring $F_e = F_m$ at equilibrium gives

$$V_H = \frac{j_x B L}{ne}.$$

The resistivity of the material in the x-direction, ρ_x is given by $\rho_x = E_x/j_x$. Similarly, we can associate a Hall resistivity $\rho_H \equiv E_y/j_x$ with the Hall voltage. The electric field in the y-direction is $E_y = F_m/e = v_x B$, giving the linear relation

$$\rho_H = \frac{B}{ne}. \tag{2.1}$$

2.2 The Quantum Hall Effect

Experiments have been done to measure the Hall effect in the layer between two semiconductors at low temperatures. In this system the charge carriers are high mobility electrons confined to a plane, and may be viewed as a 2D electron gas. Here one finds that the smooth linear behaviour of the system breaks down. As the magnitude of the magnetic field increases, the Hall resistivity forms plateaus at values

$$\rho_H = \frac{h}{fe^2} \tag{2.2}$$

where f takes the values of integers and certain fractions. This is called the quantum Hall effect (QHE). When f is an integer it is called the integer quantum Hall effect (IQHE), and when f is a fraction it is called the fractional quantum Hall effect (FQHE). In addition to this phenomenon, the longitudinal resistivity ρ_x also exhibits irregular behaviour. It is observed to drop drastically in magnitude each time a plateau is formed in the Hall resistivity, as shown in figure 2.2.

The IQHE was first predicted [13] by Ando, Matsumoto, and Uemura in 1975, and was observed not long after. In 1980 Klaus von Klitzing made the unexpected discovery [14] that the quantum Hall system is exactly quantized, which means that the plateaus always will appear at exactly $\rho_H = h/ne^2$ for integers n, independently of the material used in the experiment and the geometry of the sample. Measurements have found this to be correct to nearly one part in a billion. von Klitzing used this fact to measure the fine structure constant α, and the most precise measurements of α to date have been made from quantum Hall experiments. For his discovery, von Klitzing was awarded the 1985 Nobel Prize in Physics. The exact quantization has also allowed for

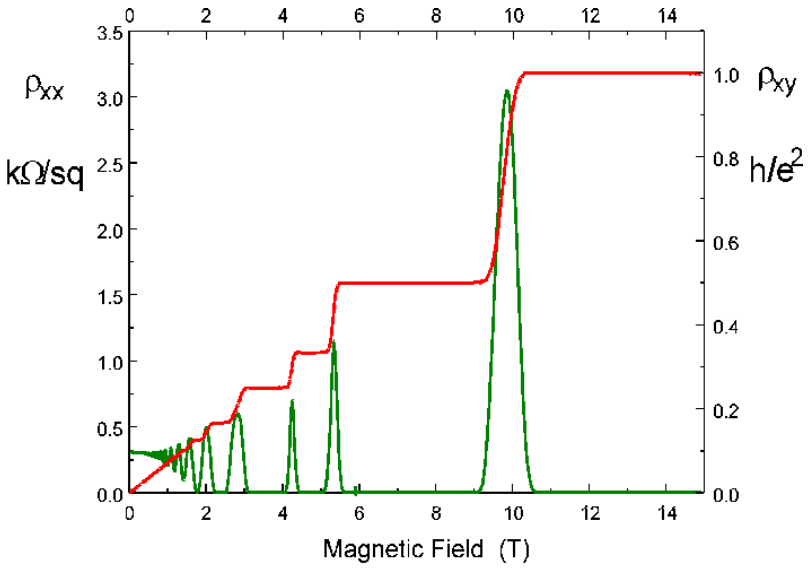

Figure 2.2: An integer QHE experiment in a GaAs-GaAlAs heterojunction at 30mK. The longitudinal resistivity ρ_{xx} is seen to drop to zero at the locations of the Hall plateaus. [12]

the definition of a new standard for resistance, namely the von Klitzing constant $R_K = h/e^2$.

In 1982, when doing experiments on the IQHE, Dan Tsui and Horst Störmer discovered the FQHE [15], for which there had been no earlier prediction. Unlike the IQHE, the FQHE could not be explained by simple qualitative arguments even though the effects appeared to be very similar in experiments. In 1983 Robert B. Laughlin proposed a set of wavefunctions [16] to explain the plateaus at the odd-denominator fractions $\nu = \frac{1}{2p+1}$, and Laughlin, Störmer and Tsui were awarded the Nobel Prize together in 1998. We will get back to the Laughlin functions in section 2.6.

The Laughlin wavefunctions did not give any clarifacation for other states though, and did not explain how the FQHE related to the integer effect. Several methods were proposed for the other fractions. Among them were the Hierarchy construction of Haldane and Halperin, and the composite fermion (CF) approach, proposed [9] by Jainendra K. Jain in 1989. In section 2.8 we will take a closer look at the composite fermions.

2.3 Landau levels

The QHE system is composed of electrons in a magnetic field. The single particle Hamiltonian for these electrons is

$$\hat{H} = \frac{1}{2m}\hat{\pi}^2 = \frac{1}{2m}\left(\hat{p} - eA(\hat{r})\right)^2 . \tag{2.3}$$

We assume the particles to be completely spin polarized by the magnetic field, giving us effectively spin-less particles and no magnetic moment interaction.

We will in the following omit the "hats" above the operators. From classical mechanics we adopt the canonical momentum $\pi = p - eA$, where A is the magnetic vector potential. We choose the symmetric gauge

$$A = \frac{1}{2} B \times r,$$

giving the components

$$A_x = -\frac{1}{2} By, \qquad A_y = \frac{1}{2} Bx.$$

In the coordinate representation, where p and A commute, the Hamiltonian becomes

$$H = \frac{1}{2m} p^2 - \frac{e}{m} A \cdot p + \frac{e^2}{2m} A^2 \tag{2.4}$$

$$= \frac{1}{2m} p^2 - \frac{e}{2m} B \cdot L + \frac{e^2}{2m} A^2, \tag{2.5}$$

since $B \times r \cdot p = B \cdot r \times p$. With the magnetic field constant along the z-axis, this Hamiltonian is just a planar harmonic oscillator with an added angular momentum term,

$$H = \frac{1}{2m}(p_x^2 + p_y^2) + \frac{1}{2} m\omega^2 (x^2 + y^2) - \omega L_z \tag{2.6}$$

where p_z has been set to zero and $\omega = \frac{eB}{2m}$. The system is quantized with the commutation relation $[x, p_x] = [y, p_y] = i\hbar$. The two-dimensional harmonic oscillator may be solved by introducing the ladder operators

$$a_+ = \sqrt{\frac{m\omega}{4\hbar}} (x - iy + \frac{i}{m\omega} p_x + \frac{1}{m\omega} p_y) \tag{2.7}$$

$$a_+^\dagger = \sqrt{\frac{m\omega}{4\hbar}} (x + iy - \frac{i}{m\omega} p_x + \frac{1}{m\omega} p_y) \tag{2.8}$$

$$a_- = \sqrt{\frac{m\omega}{4\hbar}} (x + iy + \frac{i}{m\omega} p_x - \frac{1}{m\omega} p_y) \tag{2.9}$$

$$a_-^\dagger = \sqrt{\frac{m\omega}{4\hbar}} (x - iy - \frac{i}{m\omega} p_x + \frac{1}{m\omega} p_y) \tag{2.10}$$

obeying $[a_\pm, a_\pm^\dagger] = 1$. In terms of these, the angular momentum operator is $L_z = \hbar(a_+^\dagger a_+ - a_-^\dagger a_-)$ and the Hamiltonian simplifies to

$$H = \hbar\omega(2a_+^\dagger a_+ + 1) = \hbar\omega_c (a_+^\dagger a_+ + \frac{1}{2}) \tag{2.11}$$

where the cyclotron frequency $\omega_c = 2\omega$ has been introduced for convenience. This Hamiltonian depends only on the eigenvalue n of the number operator

$a_+^\dagger a_+$, which numbers the eigenstates $|n\rangle$. The corresponding eigenvalue n_- of the operator $a_-^\dagger a_-$ can be arbitrary without affecting the energy, so each energy level is infinitely degenerate. The ground state $|0\rangle$ is defined by the relation

$$a_+|0\rangle = 0. \tag{2.12}$$

All other states are obtained by applying the raising operator $a_+^\dagger$ a number of times to the state $|0\rangle$. The ground state energy E_0 is found by multiplying (2.12) from the left with $a_+^\dagger$,

$$a_+^\dagger a_+|0\rangle = (H - \frac{1}{2}\hbar\omega_c)|0\rangle = (E_0 - \frac{1}{2}\hbar\omega_c)|0\rangle = 0,$$

giving

$$E_0 = \frac{1}{2}\hbar\omega_c.$$

If we know the energy E_n of the state $|n\rangle$, we can find the energy of $|n+1\rangle$ by using the commutator $[H, a_+^\dagger] = \hbar\omega_c a_+^\dagger$:

$$Ha_+^\dagger|n\rangle = (a_+^\dagger H + \hbar\omega_c a^\dagger)|n\rangle = (E_n + \hbar\omega_c)a_+^\dagger|n\rangle.$$

By induction from E_0, the energy of state $|n\rangle$ is then

$$E_n = \hbar\omega_c\left(n + \frac{1}{2}\right).$$

These are the energies of the so-called Landau levels (LL) at $n = 0, 1, 2, \ldots$, named after Lev D. Landau. As mentioned above, they are highly degenerate, with each Landau level including many angular momentum eigenstates. Since the angular momentum operator L_z commutes with the Hamiltonian, it is possible to diagonalize each Landau level with respect to L_z. This may be done either through a differential equation, or by an algebraic approach [17]. The solutions are, given in the complex coordinates $z = x + iy$:

$$\psi_{n,m} = N_{n,m} z^m \ell^{-m} L_n^m(\frac{z^* z}{2\ell^2}) e^{-\frac{|z|^2}{4\ell^2}}, \qquad m \geq -n, \quad n = 0, 1, 2, \ldots \tag{2.13}$$

where $\ell \equiv \sqrt{\frac{\hbar}{eB}}$ is called the magnetic length. m is the eigenvalue of L_z, and $L_{n,m}$ are the generalized Laguerre polynomials, defined by

$$L_n^m(x) \equiv \frac{x^{-m}e^x}{n!}\frac{\mathrm{d}^n}{\mathrm{d}x^n}e^{-x}x^{n+m}\ .$$

$N_{n,m}$ is a normalization constant, given by

$$N_{n,m} = \sqrt{\frac{n!}{2\pi 2^m (n+m)!}}\ .$$

In the lowest Landau level (shortened LLL) $n = 0$, and since $L_{0,m} = 1$ a basis for the LLL wave functions is

$$\psi_{0,m} = N_{0,m} z^m \ell^{-m} e^{-\frac{|z|^2}{4\ell^2}} \tag{2.14}$$

We see that LLL wave functions (sans the gaussian factor) depend only on z and not on z^*, which will be important later.

2.4 The filling factor

In an infinite system like above, each Landau level is infinitely degenerate. Experiments are of course not of infinite size, and there will be a finite number of states in each Landau level. The filling factor ν is defined as the number of filled states divided by the number of available states, which can be shown to be

$$\nu = \frac{n\phi_0}{B} \tag{2.15}$$

where n is the density of mobile charge carriers as before, and ϕ_0 is the magnetic flux quantum, defined by $\phi_0 \equiv hc/e$. This is a somewhat arbitrary definition, since the magnetic flux is not a quantized value. It is however a convenient unit in several situations. It can be shown that the Aharonov-Bohm phase of one flux quantum is exactly equal to -1, which will be important in the construction of composite fermions. If we assume a perfect Fermi sea, $\nu = 1$ would for example mean that the LLL is completely filled, and the other levels are empty.

2.5 The integer quantum Hall effect

In actual experiments, the degeneracy of the Landau levels is broken by material impurities and particle interactions. The energy spectrum transforms into a continuum, but at low temperatures this continuum is divided into energy bands formed around the previous Landau levels, as seen in figure 2.3. We will here see how this affects the Hall resistivity.

Equation (2.1) may be expressed in terms of the filling fraction as

$$\rho_H = \frac{B}{ne} = \frac{\phi_0}{\nu e} = \frac{h}{\nu e^2}. \tag{2.16}$$

Comparing with (2.2), we see that the observed values f are in fact identical to the filling factor ν. The IQHE plateaus form around filling factors $\nu = 1, 2, \ldots$, where ν levels are completely filled and the rest are empty. In figure 2.3(a) we see a state with $\nu = 2$. The Fermi energy lies in the gap between two Landau levels. Increasing the magnetic field also increases the gap between the levels, and creates new energy states due to the disorder. But at low temperatures the electrons in these new states are *localised*, meaning that they are prohibited

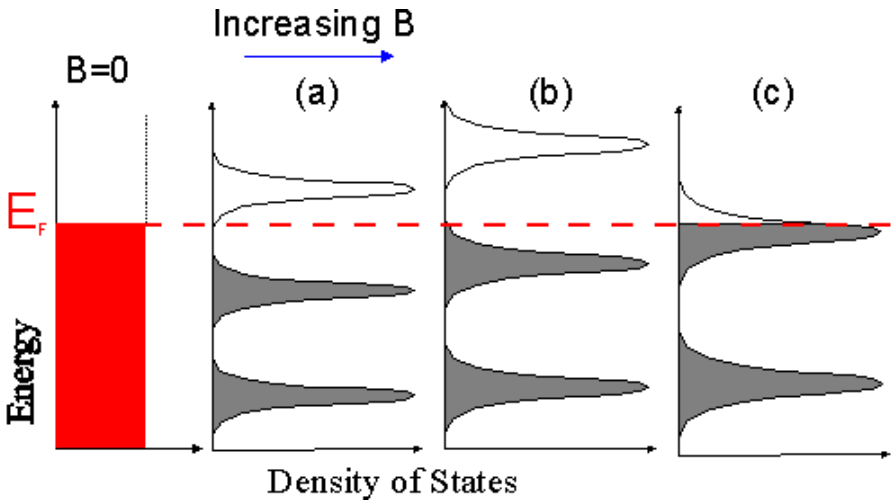

Figure 2.3: Energy bands in the QHE [12]

from extending across the sample by the impurities. Then, as we see in figure 2.3(b), the number of filled *extended* states below E_F is unchanged as long as E_F stays within the gap. This means that ρ_H is unchanged from the value it had at $\nu = 2$, thus forming a plateau.

The band gaps also explain the zeros in the longitudinal resistivity. Electric resistivity is essentially caused by interactions between the electrons and the atomic lattice, causing excitations. But at integer ν and low temperature, the thermal energy may be insufficient to excite the electrons across the band gap, and the flow becomes dissipation-less. A system like this, where the Fermi energy lies in a gap in the density of states, is called imcompressible.

2.6 The fractional quantum Hall effect

The QHE is also observed at many other filling factors p/q, where p and q are integers. The fractions that have proven easiest to describe are

$$\nu = \frac{n}{2pn \pm 1}, \tag{2.17}$$

where p and n are integers, but many other more exotic fractions are also observed. Among them are for instance the states $\nu = 4/11, 5/13, 6/15, \ldots$, which do not fit into the above sequence. Due to a particle-hole symmetry in the Landau levels, the effect is also present in the states

$$1 - \frac{n}{2pn \pm 1},$$

but we will not consider these states in this text. This effect can not be explained qualitatively in terms of single particle states as above. It is a collective phenomenon, born from the interactions between particles.

In 1983, Robert Laughlin proposed trial ground state wave funtions for the filling factors

$$\nu = \frac{1}{2p+1}. \tag{2.18}$$

These are given by

$$\psi_{1/(2p+1)} = \prod_{j<k}(z_j - z_k)^{2p+1} e^{-\sum_i |z_i|^2/4\ell^2}. \tag{2.19}$$

These wavefunctions are exact for repulsive interactions of vanishing range [18]. This can be rationalized by looking at the form of the wavefunctions. The factor $(z_j - z_k)^{2p+1}$ serves to keep the particles away from each other, since it goes to zero as $z_j \to z_k$, and the electrons are kept at an interaction-free distance. But although these wavefunctions were groundbreaking in their time, they do not give a complete description of the system. The Laughlin states fail to describe any filling factors other than $\nu = 1/m$, where m is an odd number. And they offer no qualitative description similar to the simple description of the IQHE.

Expansions of Laughlin's work were proposed to explain the missing fractions. Haldane and Halperin developed the hierarchy model [19,20], where fractionally charged quasiparticle or quasihole excitations in the Laughlin states form their own Laughlin states. These "daughter" states then produced the missing fractions. However, the construction requires a large amount of fractionally charged quasiparticles, and after only a few iterations there are more quasiparticles in the system than electrons.

2.7 The composite fermion model

A new interpretation of the FQHE and the Laughlin states was given by Jain in 1989 [9]. In his approach, the system is described in terms of a new kind of quasiparticle, called composite fermions (CF). The filling fractions (2.17) are interpreted as the IQHE of these composite fermions. A popular, but not entirely accurate picture of the CFs is as follows:

Imagine the magnetic field to be distributed among a number of flux quanta in the plane, and place a number of flux quanta in the electron coordinates. This corresponds to each electron "attaching" itself to the flux quanta and carrying them around. When most of the flux quanta have been "picked up" like this, the resulting composite particles will experience a much reduced magnetic field from the remaining quanta. This is an oversimplified picture, but it produces most of the physical consequences of interest in this thesis.

Jain's original formulation does not attach flux quanta to the particles, but vortices. But since a vortex has the same winding phase as a flux quantum the result is the same for all practical purposes. Bringing two particles with k attached vortices each halfway around each other gives an Aharonov-Bohm phase factor of $(-1)^k$. If we want the CFs to retain fermion statistics k must be

even, or in other words $k = 2p$ for integer p. The "effective" field experienced by the composite fermions is

$$B^* = B - 2pn\phi_0 \tag{2.20}$$

This field creates its own set of Landau levels for the composite fermions. These CF Landau levels have their own filling factors

$$\nu^* = \frac{n\phi_0}{|B^*|}$$

which are related to the original filling factors by

$$\nu = \frac{\nu^*}{2p\nu^* \pm 1} \tag{2.21}$$

where the negative in the denominator is chosen if the fields are antiparallell. Setting $\nu^* = n$ reproduces the fractions (2.17), and we see that the Jain states can be interpreted as the IQHE of the composite fermions.

2.8 Composite fermion wave functions

The usefulness of the CF picture stems largely from the fact that the composite fermions are weakly interacting. They can in fact often be assumed to be non-interacting, which is what we will do in this thesis. The problem can then be solved by constructing an antisymmetric many-body wavefunction ψ_{CF} from the single particle CF wavefunctions, where the composite fermions are observed to reside in Landau levels with an effective magnetic field and filling factor. This is done by putting the single particle wave functions in a Slater determinant, and multiplying with a Gaussian factor corresponding to the filling factor ν^*.

We of course want to end up with wavefunctions describing the many-body *electron* system, not the composite fermion system. But just as we can convert electrons to composite fermions by attaching vortices, we can convert them back by attaching vortices in the opposite direction to the composite fermions. This is done [21] by multiplying ψ_{CF} with ϕ_1^{2p}, where ϕ_1 is the wavefunction of one filled LL,

$$\phi_1 = \prod_{j<k}(z_j - z_k)e^{-\sum_i |z_i|^2/4\ell_1^2},$$

where ℓ_1 is the magnetic length at $\nu = 1$. Each Jastrow factor attaches one vortex to each particle, and is antisymmetric with respect to interchange of the particles. Attaching an even number of such factors then preserves statistics. The factor also goes to zero when the particles approach each other, increasing the mean distance between particles and creating a "correlation hole" around each electron. The interaction energy is therefore reduced, lowering the total energy of the state. It is then energetically favorable for the system to enter a CF phase when possible.

The final wavefunction now takes the form

$$\psi_e = \psi_{CF} \cdot \phi_1^{2p}. \tag{2.22}$$

The gaussian factors in both functions can be combined into one that corresponds to the real magnetic field B, since

$$\frac{1}{4(\ell^*)^2} + \frac{2p}{4\ell_1^2} = \frac{1}{4\ell^2}$$

by virtue of (2.20). If we now observe the special case $\nu^* = 1 \Leftrightarrow \nu = 1/(2p+1)$, we see that

$$\psi_{1/(2p+1)} = \prod_{j<k}(z_j - z_k)^{2p+1} e^{-\sum_i |z_i|^2/4\ell^2},$$

which is identical to the original Laughlin wavefunction (2.19). The Laughlin states are in other words a special case of the CF construction, corresponding to one filled CF Landau level.

The function ψ_e is in general not confined to the LLL. But at high magnitudes of B most of the physics occur within the LLL, and it can be shown [22] that the overlap to LLL subspace is very high. Various methods are therefore used to project the wavefunction onto the LLL subspace. This is desirable since LLL wavefunctions are analytical in z, and therefore easier to work with both analytically and numerically. We shall however see in chapter 4 that this projection is not necessary in the CFT approach.

2.9 The LLL projection

Omitting the Gaussian and the normalization factor for simplicity, the single-particle wavefunction (2.13) may be rewritten

$$\psi_{n,m} = \sum_{k=k_0}^{n} (-1)^k \binom{n+m}{n-k} \frac{1}{2^k k!} (z^*)^k z^{k+m} \ ,$$

where $k_0 = \max(0, -m)$. The projection to the LLL is done [21] by moving all conjugated variables to the left of the unconjugated ones, and performing the replacement

$$z_i^* \leftarrow 2\frac{\partial}{\partial z_i} \equiv 2\partial_i \ .$$

This results in the projected single-particle states $\eta_{n,m}$:

$$\eta_{n,m} = \sum_{k=k_0}^{n} (-1)^k \binom{n+m}{n-k} \frac{1}{k!} \partial^k z^{k+m} \ .$$

Substituting

$$\partial^k z^{k+m} = \sum_{\alpha=k_0}^{k} \binom{k+m}{k-\alpha} \frac{k!}{\alpha!} z^{m+\alpha} \partial^\alpha \ ,$$

and rewriting the sums using

$$\sum_{k=k_0}^{n} \sum_{\alpha=k_0}^{k} = \sum_{\alpha=k_0}^{n} \sum_{k=\alpha}^{n}$$

one can observe that the sum over k is proportional to $(1-1)^{n-\alpha}$, i.e. to $\delta_{n,\alpha}$. Thus, only the terms with $k = n = \alpha$ survive, giving the result

$$\eta_{n,m} = \frac{(-1)^n}{n!} z^{m+n} \partial^n \ . \tag{2.23}$$

The projection of (2.22) onto the LLL subspace can now be constructed by choosing a set of single-particle states $(n_1, m_1), (n_2, m_2), \ldots$ to give the desired final state. The projected single-particle wavefunctions $\eta_k \equiv \eta_{n_k,m_k}$ are put into a Slater determinant to produce the projection of ψ_{CF}, which is then multiplied with the Jastrow factor ϕ_1^{2p}, giving

$$\psi_e = \begin{vmatrix} \eta_1(z_1) & \eta_1(z_2) & \cdots \\ \eta_2(z_1) & \eta_2(z_2) & \cdots \\ \vdots & \vdots & \ddots \end{vmatrix} \prod_{j<k} (z_j - z_k)^{2p} \ .$$

Other projection methods than this one are also used, but we will not concern ourselves with these in this text. The important thing to note is that LLL wavefunctions in the CF model are written as a Slater determinant multiplied with a Jastrow factor to a certain power.

Chapter 3

Rotating Bose gases

The Bose-Einstein Condensate (BEC) is a highly unclassical phase of matter first predicted by the indian physicist Satyendra N. Bose and first described in detail by Albert Einstein. It is a consequence of the differing statistics of bosonic particles compared to their classical counterparts. As the temperature approaches zero all the particles condense in the lowest energy level, and their wavefunctions become highly overlapping. One consequence of this is the superfluidity of helium-4.

Gaseous Bose-Einstein condensates are formed by laser cooling a gas of bosonic atoms, typically alkali atoms such as rubidium. The gas is trapped in a potential, and it is further cooled by lowering the height of the potential gradually to let the most energetic particles escape. This is called evaporative cooling. The form of the potential is often that of an harmonic oscillator, but it may also have other shapes.

The BEC may be rotated by "stirring" it with a laser, or by rotating the confining potential. The system then generates quantized vortices arranged in a triangular lattice. At very high angular momenta the vortex lattice is expected to melt, and the system is said to be in the quantum Hall regime. Present-day experiments are close to achieving the rotation speeds required for this, but they are impeded by the fact that the centrifugal potential eventually cancels the trapping potential, causing the gas to fly apart. There are however suggestions on how to circumvent this problem. One of these is to add a small quartic term to the potential, another is to use an optical lattice.

At first glance it seems peculiar that this rapidly rotating Bose gas should have anything to do with the QHE. After all, the QHE is highly dependent on charged fermions moving in a magnetic field. In the BEC the particles in question are chargeless bosons, and we have no magnetic field. We shall however see that there are more similarities than meets the eye.

3.1 Bosons in an harmonic trap

We will consider a gas of spinless bosons in an isotropic three dimensional harmonic oscillator. A single particle in this potential has the hamiltonian

$$H = \frac{\mathbf{p}^2}{2m} + \frac{1}{2}m\omega^2\mathbf{r}^2 \tag{3.1}$$

$$= \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + \frac{p_z^2}{2m} + \frac{1}{2}m\omega^2x^2 + \frac{1}{2}m\omega^2y^2 + \frac{1}{2}m\omega^2z^2 \tag{3.2}$$

which is an ordinary 3-dimensional harmonic oscillator. By separating out the z-dependent terms in a transverse hamiltonian H_z and rewriting the rest, this can be expressed as

$$H = \frac{1}{2m}\left[(p_x + m\omega y)^2 + (p_y - m\omega x)^2\right] + \omega(xp_y - yp_x) + H_z \ . \tag{3.3}$$

By defining the vector field

$$\mathbf{A} = m\omega \begin{bmatrix} -y \\ x \\ 0 \end{bmatrix} \tag{3.4}$$

we can write this as

$$H = \frac{1}{2m}(\mathbf{p}_{xy} - \mathbf{A})^2 + \omega(xp_y - yp_x) + H_z \tag{3.5}$$

where $\mathbf{p}_{xy} = [p_x, p_y, 0]$. The first term is similar to the hamiltonian for electrons in a magnetic field $\mathbf{B} = \nabla \times \mathbf{A}$, and is in fact identical to the Hamiltonian (2.3) of the QHE. This motivates us to define an "effective" magnetic field,

$$\nabla \times \mathbf{A} = 2m\omega\mathbf{z} \equiv \mathbf{B}. \tag{3.6}$$

We adopt the π-operator from the last chapter. The second term in (3.5) is just an angular momentum operator. Thus we end up with the hamiltonian

$$H = \frac{1}{2m}\pi^2 + \omega L_z + H_z. \tag{3.7}$$

The operators π^2, L_z and H_z all commute with each other, and we can therefore assign independent quantum numbers to each term in the sum. We call these respectively n, m and n_z. The first term is responsible for forming effective Landau levels corresponding to the effective magnetic field. Their formation is the same as the one described in section 2.3. The quantum number n represents the Landau levels and can have the values $n = 1, 2, 3, \ldots$, while the angular momentum number m takes values $m \geq -n$. The last number n_z is the harmonic oscillator quantum number in the $\mathbf{z}$ direction.

3.2 The yrast state

It is useful here to introduce the concept of the yrast state, borrowed from nuclear physics. It is defined as the state of lowest energy E for a given angular momentum L, or equivalently the state of highest L for a given E. The word "yrast" comes from the Swedish language, and means "the most dizzy".

A single particle is in the yrast state when all the energy is angular momentum energy, $E = \omega L$. In the view of (3.5), this corresponds to setting $n = 0$ and $n_z = 0$. In a system of N such particles, the total angular momentum is

$$L = \sum m_i \ . \tag{3.8}$$

With no interaction in the system, the yrast state has energy $E = \omega L$ also in the many-body case. Since there are many ways to distribute the angular momentum L between the particles, this yrast state will be highly degenerate.

The degeneracy will be lifted if we add interaction, resulting in a new yrast state that minimizes the interaction energy. In experiments one typically has a hard core interaction, which we will represent by a delta function. We will also assume the interaction energy to be much lower than the harmonic oscillator strength ω. The gap to the first exited states of n and n_z is then considerably larger than the interaction splitting, allowing us to set $n = n_z = 0$ and still be able to study the entire interaction energy spectrum. The gas then remains two dimensional ($n_z = 0$) and in the lowest Landau level ($n = 0$). We can now omit terms from (3.5) and are left with the effective Hamiltonian

$$H = \omega L + g \sum_{i<j} \delta^2(z_i - z_j) \tag{3.9}$$

and a total energy given by $E_{tot} = E_L + E_{int}$. The assumption of low interaction energy means that $gN << \omega$.

This Hamiltonian can not be solved analytically, but we can say something about the form of the wavefunctions. Since the wavefunctions describe bosons, they must be symmetric. They should also be two-dimensional and restricted to the lowest Landau level. This means that the polynomial part of the wavefunctions must be analytical in z. A basis for this Hilbert space is

$$\psi = p(z_1, \ldots, z_N) e^{-\sum_i^N |z_i|^2/4\ell^2}$$

where p is a symmetric polynomial. Here, ℓ is the magnetic length associated with the effective magnetic field B. The gaussian factor $e^{-\sum_i^N |z_i|^2/4\ell^2}$ is common for all LLL wavefunctions, and will be omitted in the following discussion. The energy states must also be eigenstates of the angular momentum operator L_z. Any product $p = z_1^a z_2^b z_3^c \ldots$ fulfills this. The eigenvalue of p is the sum of the exponents $a+b+c+\ldots$. A sum of such terms with the same eigenvalue is called a homogeneous polynomial and is also an eigenfunction of L_z. The energy states are therefore symmetric homogeneous polynomials multiplied with a Gaussian.

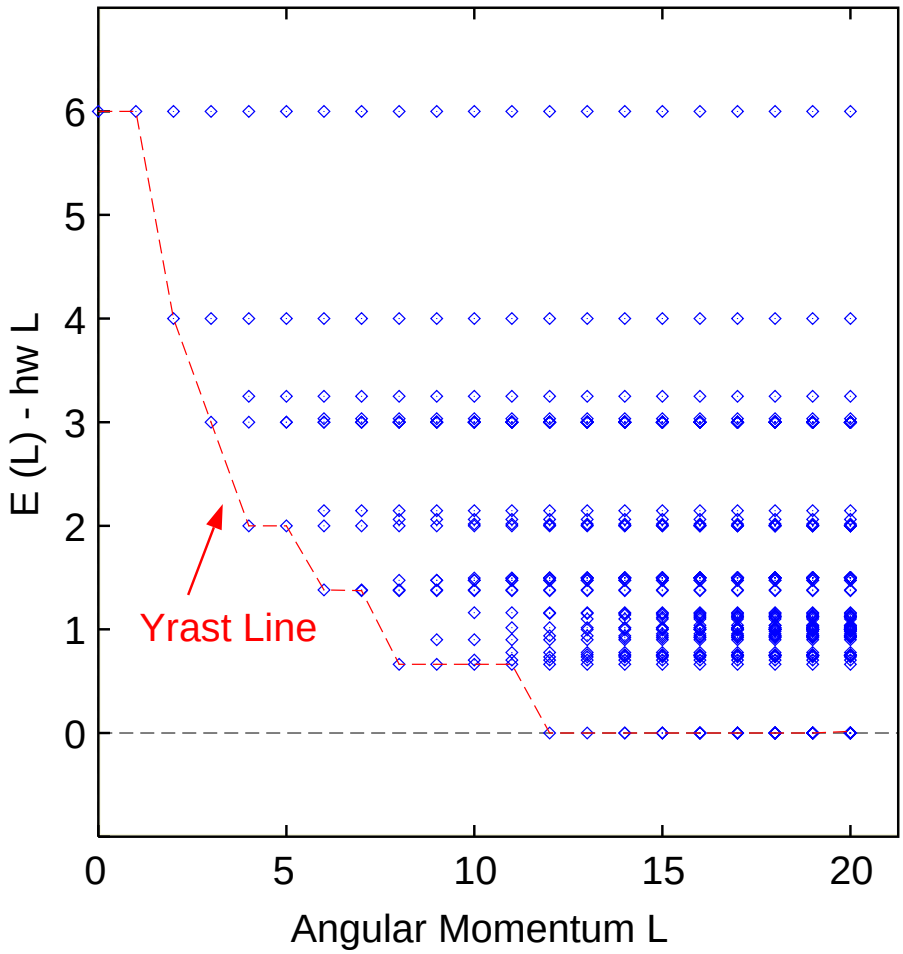

Figure 3.1: Interaction energy spectrum of (3.9) for $N = 4$ particles. [11]

3.3 The energy spectrum

The energy spectrum for the Hamiltonian (3.9) may be found through numerical diagonalization. The spectrum for $N = 4$ particles [11] is shown in figure 3.1, where the linear term ωL has been subtracted to plot only the interaction energy as a function of L. The dotted line connecting the lowermost energies at each value of L (the yrast states) is called the yrast line. At $L = 0$ the only possible state is just a regular Bose-Einstein condensate, where all the particles condense in the center of the oscillator. The wavefunction (3.2) for this state is just the Gaussian, where the polynomial p is a constant.

At $L = 1$ there is also only one possible state, namely the polynomial

$$\psi_{L=1} = z_1 + z_2 + z_3 + \ldots = \sum_i^N z_i.$$

As the angular momentum increases, the particles are allowed to spread out

in the plane. They can therefore minimize the interaction energy by keeping a distance to each other, and lower energy states become available as L increases. From $L = N(N-1)$ and onward, the yrast states have zero interaction energy, and wavefunctions can be constructed analogous to the QHE. In fact, the state at $L = N(N-1)$ has been found [10] to be exactly

$$\psi_{L=N(N-1)} = \prod_{i<j}(z_i - z_j)^2. \tag{3.10}$$

For $L > N(N-1)$ there are many ways to distribute the angular momentum among the particles for a given L, and the zero-interaction yrast states are degenerate. They all have the form

$$\psi_{L>N(N-1)} = p(z_1, z_2, \ldots) \times \psi_{L=N(N-1)} \tag{3.11}$$

where p is some symmetric homogeneous polynomial. Like in the QHE, the Jastrow factors in (3.10) keep the particles apart from each other. In this case the interaction is completely removed, since it has the form of a delta function.

3.4 Composite fermions in the BEC

The similarity between the wavefunctions (2.22) and (3.11) suggests that many of the tools from QHE physics can be used in the rotating BEC as well. In particular, it is of interest to use the CF model to describe the rotating BEC. This has been done in [10,11,23–26]. As seen above, the bosons can be viewed as moving in an effective magnetic field piercing the plane. One therefore expects them to be able to form composite fermions by attaching to an *odd* number of vortices. Analogous to section 2.7, bosonic wavefunctions can be constructed by attaching an odd number of vortices to antisymmetric CF wavefunctions. As we remember, m vortices are attached by multiplying the wavefunction with the Jastrow factor

$$J_N = \prod_{i<j}^{N}(z_i - z_j)^m.$$

But in this case, m must be odd for the final wavefunction to be symmetric.

Assuming the CFs to be non-interacting, the CF wavefunction may be constructed by putting single-particle fermion wavefunctions in a Slater determinant, as described in section 2.9.

Chapter 4

Conformal Field Theory in the Quantum Hall Effect

A connection between the FQHE and conformal field theory (CFT) has been discovered in later years. It was observed in the early '90s [27–33] that certain wavefunctions in the FQHE could be expressed as CFT correlators. Later this framework was expanded to reproduce wavefunctions obtained from Jain's composite fermion construction [34] and the hierarchy scheme of Haldane and Halperin [35]. CFT has also been used [27] to propose the so-called Pfaffian wave function for the $\nu = 1/2$ state, whose quasiholes exhibit non-Abelian fractional statistics.

We will not go deep into the large subject of CFT in this thesis. Suffice it to say that two-dimensional CFTs are scale invariant in addition to Poincaré invariant. Also, opposed to perturbative quantum field theories, CFTs rely very little on the Lagrangian or Hamiltonian formalism. They are rather characterized by an operator product algebra, which gives a set of multiplication rules for local fields. It is assumed that a product of local operators at different points can always be expressed as a linear combination of well-defined local operators, roughly

$$A(x)B(y) = \sum_i c_i(x-y)C_i(y) \ , \tag{4.1}$$

where the c_i are analytic functions. This is called an operator product expansion (OPE). For a more detailed discussion, see i.e. [36] or [37].

This chapter will be organized as follows. First we will examine the properties of free bosonic fields and their correlation functions. We will then define the important vertex operators. Lastly we will reiterate the work done in later years connecting CFT and QHE, especially the expression of Jain states as conformal field theory correlators.

4.1 The Free Bosonic Field

We will here follow the procedure of chapter 2 in [37], which starts with the quantum field theory of a free boson. This has the action

$$S[\varphi] = \int d\mathbf{x}dt\mathcal{L}(\varphi, \dot{\varphi}, \nabla\varphi) \tag{4.2}$$

$$\mathcal{L} = \frac{1}{2}\left[\frac{1}{c^2}\dot{\varphi}^2 - (\nabla\varphi)^2 - m^2\varphi^2\right] \tag{4.3}$$

where $\mathcal{L}$ is the Lagrangian density and m is a parameter we later will identify as the mass of the field. c is the speed of light, which henceforth will be set equal to 1.

At this point we only want to examine some of the general properties of this field, so we will restrict ourselves to one spacial dimension to simplify the notation. We will first quantize the theory locally by replacing this one-dimensional space with a lattice of points at positions $x_n = an$, where a is the lattice spacing and n is an integer. We can later take the continuum limit by taking $a \to 0$. We will also assume that the lattice is finite with N sites, and that it obeys periodic boundary conditions, $\varphi_N = \varphi_0$. The system is then described by the N generalized coordinates φ_n, and the Lagrangian $L = \int dx\mathcal{L}$ becomes

$$L = \sum_{n=0}^{N-1} \frac{1}{2}a\left[\dot{\varphi}_n^2 - \frac{1}{a^2}(\varphi_{n+1} - \varphi_n)^2 - m^2\varphi_n^2\right], \tag{4.4}$$

whose action tends toward (4.2) in the limits $a \to 0$ and $N \to \infty$.

Classically, one can define the canonical momentum conjugate to φ:

$$\pi_n = \frac{\partial L}{\partial\dot{\varphi}_n} = a\dot{\varphi}_n \ . \tag{4.5}$$

The Hamiltonian is then

$$H = \frac{1}{2}\sum_{n=0}^{N-1}\left[\frac{1}{a}\pi_n^2 + \frac{1}{a}(\varphi_{n+1} - \varphi_n)^2 - am^2\varphi_n^2\right]. \tag{4.6}$$

We now quantize the system by imposing the following equal-time commutators:

$$[\varphi_n, \pi_m] = i\delta_{nm} \tag{4.7}$$

$$[\pi_n, \pi_m] = [\varphi_n, \varphi_m] = 0 \tag{4.8}$$

where we have set $\hbar = 1$. We can use the discreet Fourier transform on the variables:

$$\tilde{\varphi}_k = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} e^{-2\pi i k n/N} \varphi_n$$
$$\tilde{\pi}_k = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} e^{-2\pi i k n/N} \pi_n \ . \tag{4.9}$$

The index k takes integer values from 0 to $N-1$, since $\tilde{\varphi}_{k+N} = \tilde{\varphi}_k$. And since φ_n and π_n are real, the Hermitian conjugates are

$$\tilde{\varphi}_k^\dagger = \tilde{\varphi}_{-k} \qquad\qquad \tilde{\pi}_k^\dagger = \tilde{\pi}_{-k} \tag{4.10}$$

These Fourier modes obey the commutation relations

$$\begin{aligned}\left[\tilde{\varphi}_k, \tilde{\pi_q^\dagger}\right] &= \frac{1}{N} \sum_{m,n=0}^{N-1} e^{-2\pi i (km-qn)/N} \left[\varphi_m, \pi_n\right] \\ &= \frac{i}{N} \sum_{n=0}^{N-1} e^{-2\pi i n (k-q)/N} \\ &= i\delta_{kq}\end{aligned} \tag{4.11}$$

In terms of this, the Hamiltonian (4.6) becomes

$$H = \frac{1}{2} \sum_{k=0}^{N-1} \left\{ \frac{1}{a} \tilde{\pi}_k \tilde{\pi}_k^\dagger + a \tilde{\varphi}_k \tilde{\varphi}_k^\dagger \left[m^2 + (2/a^2) \left(1 - \cos \frac{2\pi k}{N} \right) \right] \right\}. \tag{4.12}$$

This is simply the Hamiltonian for a system of N uncoupled harmonic oscillators, with freqencies

$$\omega_k^2 \equiv m^2 + \frac{2}{a^2} \left(1 - \cos \frac{2\pi k}{N} \right). \tag{4.13}$$

We now define raising and lowering operators

$$\begin{aligned} a_k &= \frac{1}{\sqrt{2a\omega_k}} \left(a\omega_k \tilde{\varphi}_k + i \tilde{\pi}_k \right) \\ a_k^\dagger &= \frac{1}{\sqrt{2a\omega_k}} \left(a\omega_k \tilde{\varphi}_k^\dagger + i \tilde{\pi}_k^\dagger \right) \end{aligned} \tag{4.14}$$

obeying the commutation rules

$$\left[a_k, a_q^\dagger \right] = \delta_{kq} \ . \tag{4.15}$$

We can then write the Hamiltonian as

$$H = \frac{1}{2}\sum_{k=0}^{N-1}(a_k^\dagger a_k + a_k a_k^\dagger)\omega_k \tag{4.16}$$

$$= \sum_{k=0}^{N-1}\left[(a_k^\dagger a_k) + \frac{1}{2}\right]\omega_k. \tag{4.17}$$

The ground state $|0\rangle$ of the system is defined by

$$a_k|0\rangle = 0 \qquad \forall\ k \tag{4.18}$$

and we obtain the complete set of energy eigenstates by acting on $|0\rangle$ with all possible combinations of raising operators:

$$|k_1, k_2, \ldots, k_n\rangle = a_{k_1}^\dagger a_{k_2}^\dagger \ldots a_{k_n}^\dagger|0\rangle \tag{4.19}$$

The k_i are not necessarily different, and as such these states are not necessarily normalized. The energy of this state is

$$E[k] = E_0 + \sum_i \omega_k \tag{4.20}$$

where $E_0 = \frac{1}{2}\sum_{k=0}^{N-1}\omega_k$ is the ground state energy.

We choose to work in the Heisenberg picture, in which the time evolution of the operators a_k is determined by the Heisenberg equation:

$$\dot{a}_k = i\,[H, a_k] = -i\omega_k a_k \tag{4.21}$$

The solution to this equation is

$$a_k(t) = a_k(0)e^{-i\omega_k t}\ . \tag{4.22}$$

This, in conjunction with (4.9) and (4.14), gives the time dependence of the generalized coordinates:

$$\varphi_n(t) = \sum_{k=0}^{N-1}\sqrt{\frac{2}{Na\omega_k}}\left[e^{i(2\pi kn/N - \omega_k t)}a_k(0) + e^{-i(2\pi kn/N-\omega_k t)}a_k^\dagger(0)\right] \tag{4.23}$$

We can now take the continuum limit by sending the lattice spacing a to zero and the number N of sites to infinity, while keeping the volume $V = Na$ constant. The continuum limit of the generalized coordinates and the conjugate momenta are

$$\varphi_n \to \varphi(x) \qquad \frac{1}{a}\pi_n \to \pi(x) = \dot{\varphi}(x) \qquad (x = na) \tag{4.24}$$

Sums over sites and Kronecker deltas become

$$a\sum_{n=0}^{N-1} \to \int dx \qquad \delta_{nn'} \to a\delta(x - x') \tag{4.25}$$

so that the canonical commutation relation (4.7) becomes

$$\big[\varphi(x), \pi(x')\big] = i\delta(x - x'). \tag{4.26}$$

By replacing the discreet Fourier index k with the physical momentum $p = 2\pi k/V$, the sums over Fourier modes and Kronecker deltas in mode indices become

$$\frac{1}{V}\sum_{k=0}^{N-1} \to \int \frac{dp}{2\pi} \qquad \delta_{kk'} \to \frac{2\pi}{V}\delta(p - p') \tag{4.27}$$

We define the continuum annihilation operator as

$$a(p) = a_k\sqrt{V} \tag{4.28}$$

with commutation relations

$$\Big[a(p), a^\dagger(p')\Big] = 2\pi\delta(p - p') \tag{4.29}$$

and the associated frequency

$$\omega(p) = \sqrt{m^2 + p^2}\,,$$

which is found by Taylor expanding the cosine function in (4.13) and observing that all terms of order 4 or higher vanish as $N \to \infty$. The field $\varphi(x,t)$ is expanded in creation and annihilation operators as

$$\varphi(x,t) = \int \frac{dp}{2\pi}\left\{a(p)e^{i(px-\omega(p)t)} + a^\dagger(p)e^{-i(px-\omega(p)t)}\right\}\,. \tag{4.30}$$

We can now sit back and examine the physical properties of this field. The simplest excited states are of the form $a^\dagger(p)|0\rangle$ with energy

$$\omega(p) = \sqrt{m^2 + p^2}. \tag{4.31}$$

This is characteristic of relativistic particles, and we interpret these elementary excitations as particles of mass m and momentum p. The states (4.19) represent a collection of independent particles whose momenta are conserved separately. Since the total energy is the sum of the energies of each particle we say that the particles are free, as they do not interact. Also, by virtue of the commutation relations (4.15), the states (4.19) are symmetric under interchange of momenta. Therefore these particles are bosons, and we give the name free bosonic field to the field φ with action (4.2).

4.2 Correlation functions

To utilize this field for calculations, we need to know the vacuum expectation value of a product of field operators:

$$\langle 0|\varphi(x_1)\varphi(x_2)\dots\varphi(x_n)|0\rangle \equiv \langle\varphi(x_1)\varphi(x_2)\dots\varphi(x_n)\rangle \tag{4.32}$$

We start by finding the two-point function $\langle\varphi(x_1)\varphi(x_2)\rangle$. We can later use Wick's theorem and normal ordering to express (4.32) in terms of these propagators.

Since we will be interested in two-dimensional field theories in the remainder of this thesis, we shall from now on restrict ourselves to the case of a free massless boson in two dimensions. Adopting covariant notation, the action (4.2) for this field is

$$S = \frac{1}{2}g\int d^2x\partial_\mu\varphi\partial^\mu\varphi, \qquad \mu = 1,2 \tag{4.33}$$

where g is a normalization constant. We now want to find

$$K(\mathbf{x},\mathbf{y}) = \langle\varphi(\mathbf{x})\varphi(\mathbf{y})\rangle. \tag{4.34}$$

The equation determining $K(\mathbf{x},\mathbf{y})$ may be obtained [37] either by solving the equations of motion for the correlation functions in the path integral formalism or by rewriting (4.33)

$$S = \frac{1}{2}\int d^2xd^2y\varphi(\mathbf{x})K^{-1}(\mathbf{x},\mathbf{y})\varphi(\mathbf{y})$$

and using a general property of gaussian integrals. Either way, the result is

$$-g\partial_x^2K(\mathbf{x},\mathbf{y}) = \delta(\mathbf{x}-\mathbf{y}) \tag{4.35}$$

We demand rotational and translational invariance, so $K(\mathbf{x},\mathbf{y})$ should only depend on the distance $r = |\mathbf{x}-\mathbf{y}|$. Integrating (4.35) over x within a disk of radius r centered on y, we find

$$1 = 2\pi g \int_0^r \rho d\rho \left(-\frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho K'(\rho)) \right) \tag{4.36}$$

$$= -2\pi g \int_0^r \left(K'(\rho) + \rho K''(\rho) \right) \tag{4.37}$$

$$= -2\pi g \left(\int_0^r d\rho K'(\rho) + rK'(r) - \int_0^r d\rho K'(\rho) \right) \tag{4.38}$$

$$= -2\pi g r K'(r) \tag{4.39}$$

This has the simple solution

$$K(r) = -\frac{1}{2\pi g} \ln r + \mathcal{C} \tag{4.40}$$

where $\mathcal{C}$ is a constant of integration, so our two-point function is

$$\langle \varphi(\mathbf{x}) \varphi(\mathbf{y}) \rangle = -\frac{1}{4\pi g} \ln(\mathbf{x} - \mathbf{y})^2 + \mathcal{C} \tag{4.41}$$

4.3 Vertex Operators

The operator product algebra characterizing our CFT is a vertex operator algebra which, not surprisingly, involves the so-called vertex operators. We will not delve into the fine details of this algebra; but we will look a little closer at the properties of the vertex operators. They will turn out to be useful in the systems we are looking at in this thesis. The interested reader is referred to [37].

When quantized on a cylinder with circumference L and written with the conformal coordinates $z = e^{2\pi(\tau - ix/L)}$ and $\bar{z} = e^{2\pi(\tau + ix/L)}$, the bosonic field (4.30) may be rewritten as

$$\varphi(z, \bar{z}) = \varphi_0 - \frac{i}{4\pi g}\pi_0 \ln(z\bar{z}) + \frac{i}{\sqrt{4\pi g}} \sum_{n \neq 0} \frac{1}{n} \left(a_n z^{-n} + \bar{a}_n \bar{z}^{-n} \right) \tag{4.42}$$

as done in chapter 6.3 in [37]. The new operators a_n and $\bar{a}_n$ obey the commutation relations

$$[a_n, a_m] = [\bar{a}_n, \bar{a}_m] = n\delta_{n,-m} \qquad [a_n, \bar{a}_m] = 0 \ . \tag{4.43}$$

The two-point function (4.41) for this field becomes

$$\langle \varphi(z, \bar{z}) \varphi(w, \bar{w}) \rangle = -\frac{1}{4\pi g} \left\{ \ln(z - w) + \ln(\bar{z} - \bar{w}) \right\} + \mathcal{C} \ . \tag{4.44}$$

The vertex operators are defined by

$$\mathcal{V}_\alpha(z,\bar{z}) =: e^{i\alpha\varphi(z,\bar{z})} : \tag{4.45}$$

where the normal ordering : : means

$$\begin{aligned}\mathcal{V}_\alpha(z,\bar{z}) = \exp\left\{ i\alpha\varphi_0 + \frac{\alpha}{\sqrt{4\pi g}} \sum_{n>0} \frac{1}{n} \left(a_{-n} z^{-n} + \bar{a}_{-n} \bar{z}^{-n} \right) \right\} \\ \times \exp\left\{ \frac{\alpha}{4\pi g} \pi_0 \ln(z\bar{z}) - \frac{\alpha}{\sqrt{4\pi g}} \sum_{n<0} \frac{1}{n} \left(a_n z^{-n} + \bar{a}_n \bar{z}^{-n} \right) \right\}\end{aligned} \tag{4.46}$$

so that the operators within each exponential commute. We will henceforth set $g = 1/4\pi$. The operator product expansion of products of these vertex operators can be calculated using the following relation for a single harmonic oscillator:

$$: e^{A_1} :: e^{A_2} :=: e^{A_1+A_2} : e^{\langle A_1 A_2 \rangle} \tag{4.47}$$

where $A_i = \alpha_i a + \beta_i a^\dagger$ is some linear combination of annihilation and creation operators. In particular,

$$: e^{a\varphi_1} :: e^{b\varphi_2} :=: e^{a\varphi_1 + b\varphi_2} : e^{ab\langle \varphi_1 \varphi_2 \rangle} \tag{4.48}$$

This gives for the vertex operators (4.46)

$$\mathcal{V}_\alpha(z,\bar{z}) \mathcal{V}_\beta(w,\bar{w}) \sim |z-w|^{2\alpha\beta}\, \mathcal{V}_{\alpha+\beta}(w,\bar{w}) + \ldots \tag{4.49}$$

It can be shown (chapter 9.1 in [37]) that the correlator of a string of such vertex operators is given by

$$\langle \mathcal{V}_{\alpha_1}(z_1,\bar{z}_1) \ldots \mathcal{V}_{\alpha_1}(z_1,\bar{z}_1) \rangle = \prod_{i<j} |z_i - z_j|^{2\alpha_i\alpha_j} \tag{4.50}$$

and is non-zero only if the following neutrality condition is satisfied:

$$\alpha_1 + \alpha_2 + \ldots + \alpha_n = 0 \tag{4.51}$$

The full vertex operator (4.46) can be decomposed into a product of holomorphic and anti-holomorphic operators, with opposite chiralities,

$$\mathcal{V}_\alpha(z,\bar{z}) = V_\alpha(z) \otimes \bar{V}_\alpha(\bar{z}) \tag{4.52}$$

where

$$V_\alpha(z) =: e^{i\alpha\varphi_1(z)} : \tag{4.53}$$

$$\varphi_1(z) = \varphi_0 - ia_0 \ln z + i\sum_{n\neq 0}\frac{1}{n}a_n z^{-n} \tag{4.54}$$

with the commutation relations

$$[a_n, a_m] = n\delta_{n+m,0}, \qquad [\varphi_0, a_0] = i\ . \tag{4.55}$$

In this formalism, we cannot write $\varphi(z,\bar{z}) = \varphi_1(z) + \bar{\varphi_1}(\bar{z})$ or $\mathcal{V}_\alpha(z,\bar{z}) = V_\alpha(z)\bar{V}_\alpha(\bar{z})$, since the zero-mode φ_0 would be overcounted. As such, $\varphi_1(z)$ is not purely holomorphic.

The holomorphic part of (4.50) is

$$\langle V_{\alpha_1}(z_1)V_{\alpha_2}(z_2)\dots V_{\alpha_n}(z_n)\rangle = \prod_{i<j}(z_i - z_j)^{\alpha_i\alpha_j}\ . \tag{4.56}$$

We here start to see why CFT is enticing to use in QHE/BEC physics. The polynomial (4.56) is strikingly similar to the Jastrow factors in the CF wave-functions. In the rest of this chapter we will see how these correlators can be used to produce trial wave functions for the QHE.

4.4 Construction of the $\nu = \frac{1}{m}$ Laughlin State

The similarity between (4.56) and the Jastrow factors in the Laughlin wave functions was beginning to be exploited in the early '90s [27–33]. We will here give a summary of the construction of Laughlin states at filling fractions $\nu = 1/m$, where m is an odd integer.

Since the free-boson Lagrangian (4.3) with the mass set to zero is invariant with respect to translations $\varphi \to \varphi + const.$, it is often convenient to compactify φ on a circle of radius R. It is assumed that all variation of the field is restricted to this circle. One may then adopt the boundary condition

$$\varphi(x + L, t) = \varphi(x, t) + 2\pi kR$$

so that the field φ winds k times as one circles once around the cylinder. When mapped to the complex plane with the conformal coordinates, this winding corresponds to vortices centered at the origin. We will henceforth set $k = 1$ and $R^2 = m$, and normalize the holomorphic vertex operators (4.53) to

$$V_1(z) =: e^{i\sqrt{m}\varphi_1(z)} : \tag{4.57}$$

$$H_{\frac{1}{m}}(\eta) =: e^{\frac{i}{\sqrt{m}}\varphi_1(\eta)} : \tag{4.58}$$

V will represent electrons, while H will represent quasiholes. We will henceforth suppress the normal ordering symbol : :, and view all vertex operators as implicitly normal ordered. The holomorphic part of (4.44) is normalized so that

$$\langle \varphi_1(z)\varphi_1(w)\rangle = -\ln(z-w) \tag{4.59}$$

The vertex operators then obey the relations

$$\begin{aligned} e^{i\alpha\varphi_1(z)}e^{i\beta\varphi_1(w)} &= e^{i\pi\alpha\beta}e^{i\beta\varphi_1(w)}e^{i\alpha\varphi_1(z)} = (z-w)^{\alpha\beta}e^{i\alpha\varphi_1(z)+i\beta\varphi_1(w)} \\ &\sim (z-w)^{\alpha\beta}e^{i(\alpha+\beta)\varphi_1(w)} \end{aligned} \tag{4.60}$$

by virtue of the Baker-Hausdorff formula and the OPE (4.49). It is easy to check that this makes the operators V_1 anticommute among themselves, appropriate for a fermionic operator.

The field φ_1 has a conserved current due to gauge invariance and Noether's theorem. This current is given by the charge density operator $J(z)$, which is normalized as

$$J(z) = \frac{i}{\sqrt{m}}\partial_z\varphi_1(z). \tag{4.61}$$

The corresponding charge operator is then given by

$$\mathcal{Q} = \frac{1}{2\pi i}\oint dz J(z) = \frac{1}{\sqrt{m}}\frac{1}{2\pi}\oint dz\partial_z\varphi_1(z) \tag{4.62}$$

where the contour encircles the whole system. The U(1) charge Q of a field $A(z)$ is defined by the commutator

$$[\mathcal{Q}, A(w)] = QA(w) \tag{4.63}$$

which for the vertex operator (4.57) gives

$$[\mathcal{Q}, V_1(w)] = \frac{1}{2\pi i}\oint dz i\partial_z\varphi_1(z)V_1(w) \tag{4.64}$$

$$= \frac{1}{2\pi i}\oint dz(V_1(w)\frac{1}{z-w} + \mathrm{Reg}(z)) \tag{4.65}$$

$$= V_1(w) \tag{4.66}$$

where $\mathrm{Reg}(z)$ stands for a term regular in z. The expression of the commutator as a contour integral can be deduced from the formalism of OPEs, while in the second line we have used the OPE $\partial\varphi_1(z)V_1(w) = -iV_1(w)\frac{1}{z-w} + \mathrm{Reg}(z)$. It is the holomorphic part of the OPE

$$\partial\varphi_1(z)\mathcal{V}_\alpha(w,\bar{w}) = \sum_{n=0}^{\infty} \frac{(i\alpha)^n}{n!} \partial\varphi(z) : \varphi(w,\bar{w})^n : \tag{4.67}$$

$$\sim \frac{1}{z-w} \sum_{n=1}^{\infty} \frac{(i\alpha)^n}{(n-1)!} : \varphi(w,\bar{w})^{n-1} : \tag{4.68}$$

$$\sim -i\alpha\mathcal{V}_\alpha(w,\bar{w}) \frac{1}{z-w} \tag{4.69}$$

as calculated in [37]. A similar calculation as (4.66) gives $Q = 1/m$ for the quasihole operator $H_{1/m}$. It should be noted that Q is not to be interpreted as the electric charge, but rather a measure of the vorticity of the system. This can be seen by looking at (4.60). Exchanging the operators introduces a phase $e^{i\pi\alpha\beta}$. The exchange is equivalent to taking one operator in a half-circle around the other, and then translating the system. This is comparable to the Aharonov-Bohm phase of a particle taken around a topological vortex, for instance in type II superconductors. The operator $H(\eta)$ introduces a positive vorticity, which corresponds to a local depletion of the electron liquid at the quasihole coordinate η. Similarly, an operator introducing a negative vorticity would correspond to a local increase in density. This will be exploited to construct operators representing quasiparticles. By convention [34], an operator adds an electron to the liquid if its argument is an electron coordinate z, while no electron is added if the argument is a quasihole coordinate η. The operator $V(z)$ then depletes the liquid locally just like H and adds an electron at the electron coordinate z, thus expanding the droplet slightly while leaving it locally intact.

The local depletion or contraction caused by an operator with U(1) charge Q leads to an excess electron number given by $\Delta n = \delta n - Q$, where δn is the number of electrons added by the operator. This makes it possible to relate the eletric charge Q_{el} of a particle to its U(1) charge Q by the relation $Q_{el} = -e\Delta n = e(Q - \delta n)$. Since $Q_{el} \propto \Delta n$, the charge density is proportional to the particle density. This will have significance in the next chapter, where the particle density will be a more suitable concept.

The $\nu = 1/m$ Lauglin wavefunctions (2.19) can now be expressed as the correlator of a series of vertex operators:

$$\Psi_L(\{z_i\}) = \langle 0|\mathcal{R}\{V_1(z_1)V_1(z_2)\dots V_1(z_N)e^{-i\sqrt{m}\rho_m \int d^2z\varphi_1(z)}\}|0\rangle \tag{4.70}$$

$$\equiv \langle V_1(z_1)V_1(z_2)\dots V_1(z_N)\rangle_{1/m} \tag{4.71}$$

$$= \prod_{i<j}(z_i - z_j)^m e^{-\sum_i |z_i|^2/4\ell^2} \tag{4.72}$$

where $\mathcal{R}$ denotes the radial ordering $|z_1| \geq |z_2| \geq \dots \geq |z_N|$. The average $\langle\dots\rangle_{1/m}$ is defined not as a pure vacuum expectation value, but implies also radial ordering and the added exponential operator $e^{-i\sqrt{m}\rho_m \int d^2z\varphi_1(z)}$. This represents a constant smeared background charge density $\rho_m = -2\pi\ell^2/m$, where ℓ is the magnetic length. It is needed because of the charge neutrality condition

(4.51), which ensures that the correlator is zero unless $N = \rho_m \int d^2z = \rho_m A$ where A is the area of the system, as shown in [34]. Amazingly, this exponential factor produces exactly the correct gaussian factor in the final wavefunction.

4.5 The quasiparticle

We have seen that the CFT approach is useful for reproducing Laughlin's wavefunctions at $\nu = 1/(2p+1)$. But what about other filling fractions? In the last few years, some exciting expansions of this framework have managed to also reproduce the Composite Fermion wavefunctions of J.K. Jain. We will in the following give a summary of this approach, which requires the construction of a quasiparticle operator.

As seen above, the positive U(1) charge of the operator H leads to a local depletion of the electron liquid which is interpreted as a quasihole. A natural guess for a quasiparticle operator is therefore to switch the sign in the exponent of (4.58) so that it becomes $e^{-\frac{1}{\sqrt{m}}\varphi_1(\eta)}$. This is however problematic, since it upon insertion into the correlator (4.70) introduces singular terms proportional to $\prod_i (z_i - \eta)^{-1}$. Instead, inspired by the composite fermion picture, Hansson et al. [34] defined a quasiparticle operator $P_{\frac{1}{m}}(z)$ with U(1) charge $Q = (1 - 1/m)$. It replaced one of the electron operators $V_1(z)$, and can thus be viewed as a modified electron operator with a different amount of vorticity. This modified electron operator is given by

$$P_{\frac{1}{m}}(z) = \partial e^{i(\sqrt{m} - \frac{1}{\sqrt{m}})\varphi_1(z)} \ . \tag{4.73}$$

This operator is a combination of an inverse quasihole with charge $-1/m$ and an ordinary electron operator with charge 1. This is inspired by the form of the Laughlin wavefunctions. The Jastrow factors set up a correlation hole around each electron, allowing the liquid to contract locally around an electron without violating the Pauli principle. This corresponds to an inverse quasihole with an electron in the middle. The charge $Q = (1 - 1/m)$ is suggestive of this, since it leads to a slightly less vorticity than the $Q = 1$ of an electron, thus contracting the liquid slightly.

With this operator, one can construct wavefunctions for any number of quasiparticles. The partial derivative in (4.73) can be shown to be necessary to produce nonzero wavefunctions.

4.6 The positive Jain series

The Jain states in the so-called positive Jain sequence occur at filling fractions $\nu = n/(2np+1)$. In the CF picture, their ground state wavefunctions are constructed as n filled Landau levels of composite fermions with $2p$ flux quanta attached. In particular, the $\nu = 2/5$ state corresponds to filling the lowest two CF Landau levels. The quasiparticles in the CF picture are composite fermions added to a CF Landau level. The $\nu = 2/5$ state may therefore be viewed as a

compact state of $N/2$ quasiparticles. For the state to be compact, the CF's in the second Landau level must have the lowest possible angular momentum.

Inspired by this, Hansson et al. [34] proposed a wavefunction for this state by constructing a correlator of $M = N/2$ quasiparticles, and considering a maximum density droplet. This resulted in an anyonic wavefunction involving a large number of derivatives. The wavefunction could be simplified by moving the derivatives all the way to the left, and by introducing a second free bosonic field $\varphi_2(z)$ that commutes with φ_1. It turned out that the extra terms resulting from the derivatives acting on more factors had little significance. The wavefunction could then be written

$$\Psi^{\mathrm{CF}}_{2/5}(\{z_i\}) = \mathcal{A}\{\langle \prod_{i=1}^{M} V_2(z_i) \prod_{j=M+1}^{2M} V_1(z_j)\rangle\} \tag{4.74}$$

where the new operator

$$V_2(z) = \partial e^{i\sqrt{\frac{4}{3}}\varphi_1(z)} e^{i\sqrt{\frac{5}{3}}\varphi_2(z)} \tag{4.75}$$

has been defined, and where m has been set to 3 for simplicity. The wavefunction (4.74) could then be shown [34] to be identical to the one obtained from the CF picture.

We have seen that V_1 can be interpreted as an electron operator, but what about V_2? They anticommute among themselves and are thus fermionic, and since the whole construction is inspired by CF picture we expect them to represent composite fermions. But since the original system is composed of electrons, we want to interpret the V_2s as composite *electron* operators. They should therefore have the same charge as V_1. This is ensured if the charge density operator is redefined as

$$J(z) = \frac{i}{\sqrt{3}}\partial\varphi_1(z) + \frac{i}{\sqrt{15}}\partial\varphi_2(z). \tag{4.76}$$

The new charge density operator sets a different condition on the background charge. Charge neutrality must still hold, and the exponential operator in (4.72) must be changed accordingly. It should also reproduce the correct exponential factor for electrons in the LLL. This is achieved by redefining the expectation value as

$$\langle \ldots \rangle_{2/5} \equiv \langle 0| \ldots e^{-i\sqrt{15}\tilde{\rho}_3 \int_{\bar{A}} d^2z \varphi_2(z)} e^{-i\sqrt{3}\rho_3 \int_A d^2z \varphi_1(z)} |0\rangle \tag{4.77}$$

where $\tilde{\rho}_3 = \frac{1}{15}\rho_0$, and the coefficients in front of the integrals are taken from the denominators in (4.76). The total background electron density is then $(\frac{1}{3} + \frac{1}{15})\rho_0 = \frac{2}{5}\rho_0$. In other words: By demanding that V_2 describes unit charge particles in the LLL, the correct charge density for the filling fraction $\nu = 2/5$ is obtained.

Further Jain states in the $p = 1$ Jain series $\nu = n/(2n + 1)$ are constructed by the same pattern: At each new level n, construct the n independent quasihole operators. It can be shown that at level n there are n possible quasihole operators, one for each level. The inverse quasihole operator in the highest level is combined with V_n to construct a quasiparticle operator, and a wavefunction for level $n - 1$ may be written as a certain number of these operators in a correlator. This motivates the construction of a new operator V_{n+1}.The charge density operator and the background charge are then adjusted to make sure the new operator describes unit charge particles in the LLL. The operators $V_{2p,n}$ for general p and n are listed in appendix A. From these one can check that a product of two operators at the same level n, $V_{p,n}(z_i)$ and $V_{p,n}(z_j)$ in the correlation function contributes to a factor $(z_i - z_j)^{2p+1}$ in the final wavefunction, and that the product of two operators at different levels $V_{p,n_1}(z_i)$ and $V_{p,n_2}(z_i)$ gives a factor $(z_i - z_j)^{2p}$.

Chapter 5

CFT description of rapidly rotating Bose gases

So far, we have seen that composite fermion wave functions in the fractional quantum Hall effect may be reproduced by the use of conformal field theory correlators. We have also seen that certain Bose gases at high angular momenta share a mathematical property with the QHE that makes it possible to utilize the same techniques on both systems. This immediately suggests that it is possible to construct wavefunctions for this system as well using CFT.

It turns out that this is possible with only a small adjustment. In this chapter, we will translate the formalism of the previous chapter to produce trial wavefunctions for rotating Bose gases.

5.1 Laughlin states

As seen in section 3.4, the formation of composite fermions in the QHE may be translated to the case of a weakly interacting Bose gas by attaching an odd number of vortices to the alkali atoms in the gas. Since the Jastrow factor associated with this flux attachment has negative parity, this changes the bosons to composite fermions. The resulting composite fermions are weakly interacting, and experience a much reduced "magnetic" field, given by

$$B^* = B - q\phi_0\rho \tag{5.1}$$

where ϕ_0 is a flux quantum analogous to the magnetic flux quantum in the QHE, and ρ is the particle density. The number of vortices q is now an odd number. The filling factors are $\nu = \frac{\rho\phi_0}{B}$ in the origianl (bosonic) Landau levels and $\nu^* = \frac{\rho\phi_0}{|B^*|}$ in the CF Landau levels. The two are related by

$$\nu = \frac{\nu^*}{q\nu^* \pm 1} \tag{5.2}$$

so that the Jain sequence in this case is

$$\nu = \frac{n}{qn \pm 1} \tag{5.3}$$

We will start with the Laughlin states. The bosonic Laughlin states occur at filling fractions $\nu = 1/m$, where m now is an even integer. The operators (4.57) and (4.58) are thus written the same way as before,

$$V_1(z) = e^{i\sqrt{m}\varphi_1(z)} \tag{5.4}$$

$$H_{\frac{1}{m}}(\eta) = e^{\frac{i}{\sqrt{m}}\varphi_1(\eta)}, \tag{5.5}$$

where normal ordering is implied. But in this case, the operators V_1 commute among themselves, and we interpret them as representing bosonic atoms. The charge density operator is again normalized as

$$J(z) = \frac{i}{\sqrt{m}}\partial_z\varphi_1(z),$$

so that the operators V_1 have U(1) charge 1. Again, this is not the electric charge, but has the interpretation of vorticity. In fact, since the atoms in the gas have zero electric charge, we will rather stick to its counterpart, the particle number, so that the background charge may be viewed as a background particle density. The Laughlin wavefunctions are now written the same way as (4.72), namely

$$\begin{aligned}\Psi_L(\{z_i\}) &= \langle V_1(z_1)V_1(z_2)\dots V_1(z_N)\rangle_{1/m} \\ &= \prod_{i<j}(z_i - z_j)^m e^{-\sum_i |z_i|^2/4\ell^2},\end{aligned} \tag{5.6}$$

where the only difference is the value of m. In particular, by setting $m = 2$ the wavefunction (5.6) is identical to the yrast state (3.10) at $L = N(N-1)$. This state may thus be viewed as non-interacting composite fermions comprised of bosons with $q = m - 1 = 1$ vortex attached to each.

5.2 The $\nu = 2/3$ Jain state

Following the method proposed in [34], we would like to construct a quasiparticle operator for the bosonic system. This can be done by contracting the gas locally around one particle, producing the quasiparticle operator

$$P_{\frac{1}{m}}(z) = \partial e^{i(\sqrt{m}-\frac{1}{\sqrt{m}})\varphi_1(z)} \ . \tag{5.7}$$

This operator has charge $Q = (1 - 1/m)$, leading to a local contraction of the liquid. The wavefunction for a single quasiparticle is written as

$$\Psi^{(l)}_{1qp}(\{z_i\}) = \mathcal{S}\{e^{-|z_1|^2/4m\ell^2}\langle P_{\frac{1}{m}}(z_1)V_1(z_2)\dots V_1(z_N)\rangle\} \tag{5.8}$$

$$= \sum_i (-1)^i e^{-\sum_i |z_i|^2/4\ell^2} \prod_{j<k}^{(i)} (z_j - z_k)^m \partial_i \prod_{l\neq i} (z_l - z_i)^{m-1} \tag{5.9}$$

where $\mathcal{S}$ denotes symmetrization of the coordinates. It is calculated by evaluating the correlator and factoring out a full antisymmetric Jastrow factor. The remaining part must then also be antisymmetric to produce a symmetric wavefunction, and one can observe that this antisymmetrization has the form of a Slater determinant expanded by the first row. The quasiparticle operator (5.7) introduces an exponential factor $e^{-|z_i|^2(1-1/m)/4\ell^2}$, so the prefactor $f_1(z_i) = e^{-|z_i|^2/4m\ell^2}$ is inserted to make sure the overall gaussian factor is $e^{-\sum_j |z_j|^2/4\ell^2}$, appropriate for a LLL wavefunction.

We should now be able to construct Jain states at level n. In particular, the state at level two of the construction with $n = 2$ and $q = 1$ corresponds to the filling fraction $\nu = 2/3$. In the CF picture, this is a state with two filled LLs of composite fermions with one vortex attached. We therefore expect it to be written as a correlator with $M = N/2$ quasiparticles. Generalizing (5.7) to M quasiparticles results in an anyonic wavefunction with a compensating prefactor. This wavefunction involves many different Jastrow factors, which motivates the introduction of a new bosonic field φ_2 that obeys the same OPE as φ_1. The anyonic factors in the wavefunction may be combined into holomorphic factors if the derivatives are moved all the way to the left. This also makes the wavefunction identical to the CF wavefunction, as we will see below. The correction to the wavefunction due to this adjustment is expected to make little difference. In fact, the wavefunction for M quasiparticles is itself an ansatz, and the new wavefunction may thus be viewed as a different approximation to the same state.

The new wavefunction may be written with a new operator V_2, which includes the field φ_2, and which also inludes the partial derivatives. This new operator looks like

$$V_2(z) = \partial e^{i\sqrt{\frac{1}{2}}\varphi_1(z)} e^{i\sqrt{\frac{3}{2}}\varphi_2(z)} .$$

The CF wavefunction is then written

$$\Psi_{2/3}(\{z_i\}) = \mathcal{S}\{\langle \prod_{i=1}^{M} V_2(z_i) \prod_{j=M+1}^{2M} V_1(z_j)\rangle\} \tag{5.10}$$

where $\mathcal{S}$ denotes symmetrization.

The new operator V_2 commutes with itself, and should have unit charge. We therefore redefine the charge density operator as

$$J(z) = \frac{i}{\sqrt{2}}\partial\varphi_1(z) + \frac{i}{\sqrt{6}}\partial\varphi_2(z). \tag{5.11}$$

The background charge must then also be redefined to produce the correct gaussian in the wavefunction. Following the same procedure as in section 4.6, this results in a total background particle density $(\frac{1}{2}+\frac{1}{6})\rho_0 = \frac{2}{3}\rho_0$, as expected for the $\nu = 2/3$ state.

Evaluating the correlators in (5.10) results in the wavefunction

$$\Psi^{\mathrm{CFT}}_{2/3}(z_i) = \sum_{i_1<i_2\ldots i_M} (-1)^{\sum_k^M i_k} \partial_{z_{i_1}}\partial_{z_{i_2}}\ldots\partial_{z_{i_M}} \prod_{k<l}^{M}(z_{i_k}-z_{i_l})^2 \prod_{k_1}^{(i_2,i_3\ldots i_M)}(z_{k_1}-z_{i_1})^1 \tag{5.12}$$

$$\prod_{k_2}^{(i_1,i_3\ldots i_M)}(z_{k_2}-z_{i_2})^1\ldots\prod_{k_N}^{(i_1,i_2\ldots i_M)}(z_{k_N}-z_{i_n})^1 \prod_{m<n}^{(i_1,i_2\ldots i_M)}(z_m-z_n)^2\,.$$

This function can be shown to be identical to the wavefunction obtained from the CF picture by factoring out one antisymmetric Jastrow factor and writing the remainder as a Slater determinant, following the proof given in appendix B in [34]. The wavefunction is then the product of two antisymmetric factors, which results in a symmetric wavefunction as expected for bosons.

5.3 General Jain states

Jain states at higher levels may now be constructed with new quasiparticle operators. For this we need quasihole operators, but the operator (5.5) is no longer sufficient, since it gives non-holomorphic electron wavefunctions. For instance, the combination $V_2(z)H_{1/2}(\eta)$ gives a factor $(z-\eta)^{1/2}$ in the final wavefunction. New quasihole operators must therefore be constructed, and it can be shown that we get one quasihole operator for each level. They will include the second Bose field φ_2, and can be found from the requirements that the power of the correlator between any quasihole operator and either V_1 or V_2 should be a non-negative integer, and that they should not be expressible as a product of other quasihole or vertex operators. In the $\nu = 2/3$ state, this leads to the following two quasihole operators:

$$\begin{aligned} H_{01} &= e^{i\frac{2}{\sqrt{6}}\varphi_2(\eta)} \\ H_{10} &= e^{\frac{i}{\sqrt{2}}\varphi_1(\eta)-\frac{i}{\sqrt{6}}\varphi_2(\eta)}\,. \end{aligned} \tag{5.13}$$

It can be shown [34] that inserting H_{10} in the correlator (5.10) corresponds to a quasihole in the lowest CF Landau level in the CF picture, and that H_{01} likewise corresponds to a quasihole in the second CF LL.

A quasiparticle operator in the $\nu = 2/3$ state is constructed by combining V_2 with the quasihole operator at the highest level, i.e. H_{01}. The state at $\nu = 3/4$ is then written with the new operator V_3, and this state has its own quasihole operators. The construction then goes on recursively, and the vertex operators at a general level n are listed in appendix A.

The above construction can reproduce all the bosonic Jain states analogous to the fermionic states, i.e. with $q = 2p-1$. There are however some limitations

to the construction. It only covers the positive Jain sequence $f = \frac{n}{qn+1}$. This excludes the states at $f = \frac{n}{qn-1}$, which in the bosonic case leads to states with the filling fractions

$$\nu = \frac{n}{n-1} = \text{“}\infty\text{”}, 2, \frac{3}{2}, \frac{4}{3}, \frac{5}{4}, \ldots$$

Note that these states still reside in the LLL while having $\nu > 1$. This means that the LLL is supersaturated. While the Pauli principle forbids such fillings in the fermionic case, there is no such restriction for bosons. However, the form of the wavefunctions obtained by the CFT construction is an impediment to constructing wavefunctions for such states. The wavefunctions all contain terms of the form $(z_i - z_j)^q$, which goes to zero as the particles approach each other. This works as an effective Pauli principle, which is incompatible with a supersaturated state. This is not surprising, since the purpose of the Jastrow factors in QHE wavefunctions in the beginning was exactly to keep particles at an interaction-free distance.

The similarity of the construction to the CF picture is however suggestive of a more general possible construction. A CF state with n filled CF Landau levels may be reproduced by vertex operators at n levels of the construction. This suggests that more general compact states may be expressed by constructing a correlator with one vertex operator for each composite fermion, where V_n is used for a CF in the n'th CF LL. In particular, this idea might be exploited to approach a CFT description of the negative Jain sequence, which corresponds to quasihole condensates [38]. This could be of interest in further studies.

5.4 Hierarchy states

A different approach to constructing states with fractional filling fractions in the QHE was proposed by Haldane and Halperin [19,20]. The scheme works as follows: In the same way as the electrons condense to form the Laughlin states at $\nu_1 = 1/t_1$ for $t_1 = 1, 3, 5, \ldots$, the quasihole and quasiparticle excitations in these states may condense to form new fractional states at $\nu = 1/(t_1 \pm 1/t_2)$ for $t_2 = 2, 4, 6, \ldots$. The quasiparticles and quasiholes in these new states may then condense in the same manner and produce yet new states, and so on and so forth. This produces unique QH states at each rational filling factor $\nu = p/q \leq 1$, where q is odd. These filling factors are represented by the continued fractions

$$\nu = \cfrac{1}{t_1 + \cfrac{1}{\alpha_2 t_2 + \cfrac{1}{\ddots \cfrac{}{\alpha_n t_n}}}}$$

denoted by $\{t_1, \alpha_2 t_2, \alpha_3 t_3, \ldots \alpha_n t_n\}$, where $n = 1, 2, \ldots$ is the number of condensates. The density of the condensate at level i is $1/t_i$. α_i is equal to $+1$ if condensate i consists of quasiholes and -1 if it consists of quasiparticles.

The Jain sequence is a subset of these fractions, obtained by setting $\alpha_i t_i = -2$ for $i = 2, \dots n$, so that $\nu_n = n/((t_1 - 1)n + 1)$. If t_1 is odd this is the original Jain sequence $f = n/(2kn+1)$, and if t_1 is even we get the bosonic Jain sequence (5.3). Recent work [35] has produced candidate wave functions for all QH hierarchy states that are obtained by succesive condensation of quasielectrons, using conformal field theory. We shall here see that this may be done also for the rotating Bose gas.

The construction starts with the Laughlin states (5.6), constructed with the same operator V_1. This leads to a state with density $\nu = 1/m$. We can go to the next level of the hierarchy by introducing a new bosonic field φ_2, and construct a new vertex operator V_2 that should reflect the density of the quasiparticle condensate at the new level. This is done by expanding V_1 with factors that make the polynomials in the resulting wavefunction analytical when the correlator is calculated. It can be shown that this new operator has to include a partial derivative to produce non-zero wavefunctions.

The new operator should also be bosonic, which sets restrictions on the allowed coefficients in the exponents of V_2. The wavefunction may then be found by combining a number of these two operators in a correlator, where two operators at the same level of the hierarchy together result in a certain Jastrow factor, and two operators at different levels result in a Jastrow factor with a different exponent. This process may be repeated up to the n'th level of the hierarchy, with a new operator with another partial derivative introduced at each level.

This results in a series of vertex operators defined recursively:

$$V_{\alpha+1} = \partial V_\alpha e^{-i\varphi_\alpha/\gamma_\alpha} e^{i\gamma_{\alpha+1}\varphi_{\alpha+1}} \tag{5.14}$$

for $\alpha = 1, 2, \dots n-1$, where $\gamma_1 = \sqrt{m}$. These operators should not be confused with the operators listed in appendix A, even though the indexing is similar. These vertex operators obey the OPEs

$$\begin{aligned} V_\alpha(z)V_\alpha(w) &\sim (z-w)^{s_\alpha} \\ V_\alpha(z)V_\beta(w) &\sim (z-w)^{s_{\alpha\beta}} \end{aligned} \tag{5.15}$$

where we have suppressed the derivatives. Inserting (5.14) into these OPEs gives

$$\begin{aligned} s_{\alpha+1} &= s_\alpha + \gamma_\alpha^{-2} + \gamma_{\alpha+1}^2 - 2 \\ s_{\alpha\beta} &= s_{\beta\alpha} = s_\alpha - 1, \qquad \text{for } \beta > \alpha \end{aligned} \tag{5.16}$$

All operators V_α should be bosonic. This is achieved if all s_α are even positive integers, which implies that $\gamma_\alpha^{-2} + \gamma_{\alpha+1}^2$ is an even positive integer $t_{\alpha+1}$

$$\gamma_{\alpha+1} = \sqrt{t_{\alpha+1} - \gamma_\alpha^{-2}}, \qquad t_{\alpha+1} = 2, 4, 6, \dots \tag{5.17}$$

Combining the relations (5.17) and (5.16) now gives us the following relation for the exponents in the OPE (5.15):

$$s_\alpha = s_{\alpha\beta} + 1 = \sum_{\lambda=1}^{\alpha} t_\lambda - 2(\alpha - 1), \quad \beta > \alpha \tag{5.18}$$

Just like in the Jain states, two operators at the same level α combined results in factors $(z-w)^{s_\alpha}$ in the wavefunction, while two operators at different levels give rise to factors $(z-w)^{s_\alpha - 1}$.

The wave function is now constructed by dividing the particles into n subsets and writing the conformal correlator

$$\Psi = \mathcal{S}\langle \prod_{\alpha=1}^{n} \prod_{i_\alpha=1}^{M_\alpha} V_\alpha(z_{i_\alpha}) \rangle \tag{5.19}$$

where $\mathcal{S}$ denotes symmetrization and M_α are the number of particles in subset α. An appopriately chosen background particle density is implicit.

As an example, let us look at the state at $\nu = 4/7$. This is a fraction that does not fit into the Jain sequence. It may be written in the form of the continued fraction

$$\nu = \frac{1}{2 - \frac{1}{4}} \ ,$$

i.e. with $t_1 = 2$ and $t_2 = 4$. Looking at the relation (5.17), this means that $\gamma_1 = \sqrt{2}$ and $\gamma_2 = \sqrt{\frac{7}{2}}$. The appropriate vertex operators are then

$$V_1(z) = e^{i\sqrt{2}\varphi_1(z)} \tag{5.20}$$

$$V_2(z) = \partial e^{i\frac{1}{\sqrt{2}}\varphi_1(z)} e^{i\sqrt{\frac{7}{2}}\varphi_2(z)} \ . \tag{5.21}$$

From relation (5.16) we find $s_1 = 2$, $s_2 = 4$ and $s_{12} = 1$, and the wavefunction is

$$\Psi = \mathcal{S}\langle \prod_{\alpha=1}^{n} \prod_{i_\alpha=1}^{M_\alpha} V_\alpha(z_{i_\alpha}) \rangle \tag{5.22}$$

$$= \mathcal{S}\{ \prod_{i,j \in M_1} (z_i - z_j)^2 \prod_{i_2=1}^{M_2} \partial_{z_{i_2}} \prod_{i,j \in M_2} (z_i - z_j)^4 \prod_{i_1=1}^{M_1} \prod_{i_2=1}^{M_2} (z_{i_1} - z_{i_2})^1 \}. \tag{5.23}$$

The values of M_1 and M_2 are found by requiring the highest power of each subset to be equal, which corresponds to the two different quasiparticle condensates having the same size. This gives $M_1 = 3M_2$. Comparing this wavefunction with the one obtained for the $\nu = 2/3$ state, we see that it is slighty less dense due to the higher powers of the Jastrow factors. This is as expected, since $\frac{4}{7} < \frac{2}{3}$.

Using this method at level n of the hierarchy gives wavefunctions with filling factors

$$\nu_n = \cfrac{1}{t_1 - \cfrac{1}{t_2 - \cfrac{1}{\ddots \cfrac{}{t_{n-1} - \cfrac{1}{t_n}}}}} \tag{5.24}$$

Some remarks are in order here. This bosonic hierarchy construction deviates from the fermionic version only in the values of t_1, or in other words which Laughlin states form the starting point for the hierarchy. Starting from bosonic Laughlin states ($t_1 = s_1$ even) gives bosonic hierarchy states, while starting from fermionic Laughlin states ($t_1 = s_1$ odd) gives fermionic hierarchy states. In both cases, all s_α for $\alpha \geq 2$ must be even so that the operators V_α retain the same statistical properties as V_1. The physical properties of the two approaches are however very different.

The CFT construction described here is only appropriate for creating hierarchy states built from quasiparticle condensation and wavefunctions for condensates involving quasiholes are not obtained. The quasiparticle excitations are interpreted as local contractions of the quantum liquid. In the fermionic case, this works fine when hierarchy states are built from the Laughlin states at $\nu = \frac{1}{3}, \frac{1}{5}, \ldots$, but we cannot build quasiparticle condensates from the $\nu = 1$ Laughlin state. In this state, all the energy eigenstates in the LLL are filled, and the Pauli exclusion principle prevents the liquid from contracting further. This can also be seen from (5.24), since setting $t_1 = 1$ would give rise to filling factors $\nu > 1$, which are forbidden for the fermions in the QHE.

The bosonic counterpart of this is to set $t_1 = 0$. Here we do not come in conflict with the Pauli principle, but it would lead to negative filling factors, which is unphysical. The reason is that such states would have to be built from the (bosonic) Laughlin state "$\nu = \infty$", i.e. the state where all the bosons condense into the same energy eigenstate at $l = 0$. In this case it is not possible to create quasiparticle excitations, since the droplet is already maximally contracted.

Chapter 6

Conclusions

The main purpose of this thesis has been to explore how conformal field theory techniques may be used on the system of a rotating Bose gas. We have used CFT correlators to propose wavefunctions for several states expected to emerge as experiments achieve the angular momenta necessary for the system to enter the quantum Hall regime.

Some of the wavefunctions obtained from the CFT approach are identical to previously proposed wavefunctions obtained from numerical studies and the composite fermion approach. This reinforces the belief that composite fermion methods are suitable for the rapidly rotating Bose gas. We have also proposed wavefunctions inspired by the hierarchy construction, which predicts states also at filling fractions not covered by the Jain sequence.

The CFT approach is able to reproduce states in the positive Jain sequence and states that fit in the hierarchy construction. Common for these states is that they are built from quasiparticle condensates, but we do not obtain states involving quasihole condensates. The construction suggests however that it is possible to construct more general compact states, among them the negative Jain sequence.

Appendix A

Vertex operators in the Jain sequence

The vertex operators that reproduce the CF wavefunctions for $\nu = \frac{q}{qn+1}$ are

$$
\begin{aligned}
V_{q,1}(z) &= e^{i\sqrt{q+1}\varphi_1(z)} \\
V_{q,2}(z) &= \partial e^{i\frac{q}{\sqrt{q+1}}\varphi_1(z)} e^{i\sqrt{1+\frac{q}{q+1}}\varphi_2(z)} \\
V_{q,3}(z) &= \partial^2 e^{i\frac{q}{\sqrt{q+1}}\varphi_1(z)} e^{i\frac{q}{\sqrt{(q+1)(2q+1)}}\varphi_2(z)} e^{i\sqrt{1+\frac{q}{2q+1}}\varphi_3(z)} \\
&\dots \\
V_{q,n}(z) &= \partial^{n-1} e^{i\frac{q}{\sqrt{q+1}}\varphi_1(z)} e^{i\frac{q}{\sqrt{(q+1)(2q+1)}}\varphi_2(z)} \dots e^{i\frac{q}{\sqrt{[q(n-2)+1][q(n-1)+1]}}\varphi_{n-1}(z)} e^{i\sqrt{\frac{nq+1}{(n-1)q+1}}\varphi_n(z)}.
\end{aligned}
\tag{A.1}
$$

where $q = 2p-1$ for the rotating Bose gas and $q = 2p$ for the QHE [34].

The groundstate CF wavefunctions for $N = nM$ particles, where M is the number of quasiparticle excitations, are then written as

$$
\Psi^{CF}_{q,n}(z_i) = \mathcal{S}\{\langle \prod_{i=1}^{M} V_{q,n}(z_i) \prod_{j=M+1}^{2M} V_{q,n-1}(z_j) \dots \prod_{j=(n-1)M+1}^{nM} V_{q,1}(z_j)\rangle\}. \tag{A.2}
$$

The corresponding charge operator is

$$
J(z) = \frac{i}{\sqrt{q+1}}\partial\varphi_1(z) + \frac{i}{\sqrt{(q+1)(2q+1)}}\partial\varphi_2(z) + \ldots + \frac{i}{\sqrt{(q(n-1)+1)(qn+1)}}\partial\varphi_n(z). \tag{A.3}
$$

The coefficients in the added exponential factors due to to the background particle density are taken from the denominators of the charge density operator (A.3). Using the sum formula

$$
\sum_{k=1}^{n} \frac{1}{(q(k-1)+1)(qk+1)} = \frac{n}{qn+1} \tag{A.4}
$$

we get the correct total background particle density $\frac{n}{qn+1}\rho_0$ for the Jain states.

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